

Bounds on s -distance sets with strength t

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Abstract

A finite set X in the Euclidean unit sphere is called an s -distance set if the set of distances between any distinct two elements of X has size s . We say that t is the strength of X if X is a spherical t -design but not a spherical $(t + 1)$ -design. Delsarte-Goethals-Seidel gave an absolute bound for the cardinality of an s -distance set. The results of Neumaier and Cameron-Goethals-Seidel imply that if X is a spherical 2-distance set with strength 2, then the known absolute bound for 2-distance sets is improved. This bound are also regarded as that for a strongly regular graph with the certain condition of the Krein parameters.

In this paper, we give two generalizations of this bound to spherical s -distance sets with strength t (more generally, to s -distance sets with strength t in a two-point-homogeneous space), and to Q -polynomial association schemes. First, for any s and $s - 1 \leq t \leq 2s - 2$, we improve the known absolute bound for the size of a spherical s -distance set with strength t . Secondly, for any d , we give an absolute bound for the size of a Q -polynomial association scheme of class d with the certain conditions of the Krein parameters.

Key words: absolute bound, cometric association scheme, Q -polynomial association scheme, two-point-homogeneous space, few distance set, s -distance set, t -design, strongly regular graph.

1 Introduction

Delsarte, Goethals and Seidel [7] introduced the concept of spherical t -designs. Let $\text{Harm}_i(\mathbb{R}^m)$ be the linear space of all real homogeneous harmonic polynomials of degree i , with m variables. A finite subset X in the Euclidean unit sphere S^{m-1} is called a spherical t -design if $\sum_{x \in X} \varphi(x) = 0$ for any $\varphi \in \text{Harm}_i(\mathbb{R}^m)$ and any $1 \leq i \leq t$. We say that t is the strength of X if X is a spherical t -design but not a spherical $(t + 1)$ -design. We have an absolute lower bound for the cardinality of a spherical $2l$ -design in S^{m-1} , namely

$$|X| \geq h_0 + h_1 + \cdots + h_l,$$

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where $|\ast|$ denotes the cardinality, and $h_i = h_{i,m} = \dim \text{Harm}_i(\mathbb{R}^m)$. Indeed, $h_{i,m} = \binom{m+i-1}{i} - \binom{m+i-3}{i-2}$, $h_{0,m} = 1$, and $h_{1,m} = m$.

The size of the set of distances between distinct two elements of $X \subset S^{m-1}$ is one of important parameters of spherical codes. A finite $X \subset S^{m-1}$ is called an s -distance set if $|A(X)| = s$, where $A(X) := \{(x, y) \mid x, y \in X, x \neq y\}$ and (\cdot, \cdot) denotes the usual inner product. An absolute upper bound for an s -distance set is

$$|X| \leq h_0 + h_1 + \cdots + h_s.$$

We have the inequality $t \leq 2s$ for a spherical s -distance set with strength t [7]. One of fundamental results related with the theory of association schemes is that if $t \geq 2s - 2$, then X carries a Q -polynomial association scheme of class s [7].

A spherical 2-distance set with strength at least 2 carries an association scheme of class 2, that is, a strongly regular graph. Conversely, a strongly regular graph is embedded to the unit sphere as a 2-distance set with strength at least 2 faithfully (*i.e.* the map is injective) [2].

Cameron, Goethals and Seidel [2] proved that if the Krein parameter $q_{1,1}^1$ of a strongly regular graph is not equal to 0, then the spherical embedding with respect to the primitive idempotent E_1 has strength 2 as a design in S^{m_1-1} , where m_1 is the rank of E_1 .

On the other hand, Neumaier [14] proved that if $q_{1,1}^1 \neq 0$, then the size of the vertex set of the strongly regular graph is bounded above by $m_1(m_1 + 1)/2 = h_{0,m_1} + h_{2,m_1}$.

These two results, due to Cameron-Goethals-Seidel and Neumaier, imply that if $X \subset S^{m-1}$ is a spherical 2-distance set with strength 2, then $|X| \leq h_{0,m} + h_{2,m}$. Remark that the known absolute bound for a 2-distance set is improved because of the assumption of t -designs.

We have several examples attaining the bound $|X| \leq h_{0,m_1} + h_{2,m_1}$, namely triangular graphs [10, 4, 17], Chang graphs [3], and the strongly regular graphs obtained from the regular two-graph on 276 vertices [8, 9].

In this paper, we give two generalizations of this bound to s -distance sets with strength t in S^{m-1} (more generally, a two-point-homogeneous space), and to Q -polynomial association schemes of class s . We prove the following bounds.

Let X be a spherical s -distance set with strength $2s - i$ where $2 \leq i \leq s + 1$. Then,

$$|X| \leq \sum_{k=0}^s h_k - h_{s-i+1}. \quad (1.1)$$

When $s = 2$ and $i = 2$, this bound coincides with $|X| \leq h_0 + h_2$.

Let (X, \mathcal{R}) be an s -class Q -polynomial scheme with respect to the ordering E_0, E_1, \dots, E_s . Define $l = \max\{k \in \{0, 1, \dots, s\} \mid q_{1,0}^0 = q_{1,1}^1 = \cdots = q_{1,k}^k = 0\}$. If $(s-1)/2 \leq l \leq s-1$ holds, then

$$|X| \leq \sum_{i=0}^s h_{i,m_1} - \sum_{\substack{i=2l-s+3 \\ i \equiv s-1 \pmod{2}}}^{s-1} h_{i,m_1}. \quad (1.2)$$

If $l \leq (s - 2)/2$ holds, then

$$|X| \leq \sum_{\substack{i=0 \\ i \equiv s \pmod{2}}}^s h_{i, m_1}. \quad (1.3)$$

When $s = 2$ and $l = 0$, this bound coincides with $|X| \leq h_{0, m_1} + h_{2, m_1}$.

We have an example attaining the above bounds. There exists a finite subset X of the minimum vectors of the Leech lattice in \mathbb{R}^{24} , such that $|X| = 2025$, X is in a 22-dimensional affine subspace, and X is a spherical 3-distance set with strength 4 in S^{21} after rescaling the norm to 1. Then $|X| = h_0 + h_1 + h_3$, and hence X attains the bound (1.1). The finite set X carries a Q -polynomial association scheme of class 3 because $t = 2s - 2$ holds. Then, $l = 1$ and X attains the bound (1.2).

We also prove a similar upper bound for antipodal spherical s -distance sets with strength t . The vertex set of the dodecahedron is an antipodal spherical 5-distance set with strength 5 in S^2 , and attains the upper bound.

2 Preliminaries

2.1 Two-point-homogeneous spaces

In this section, we introduce the concept of two-point-homogeneous spaces [5, Chapter 9], [11].

Let M be a compact metric space with a distance ρ on it, and τ is the certain function in ρ (i.e. $\tau(x, y) = F(\rho(x, y))$). We call M a two-point-homogeneous space if there exists a group G acting on M such that the following assumption hold: For any $x, x', y, y' \in M$, we have $\tau(x, y) = \tau(x', y')$ if and only if there is an element $g \in G$ such that $g(x) = x'$ and $g(y) = y'$.

Let μ be the Haar measure, which is invariant under G . We assume that μ is normalized so that $\mu(M) = 1$. Let $L^2(M)$ denote the vector space of complex-valued functions u on M , satisfying

$$\int_M |u(x)|^2 d\mu(x) < \infty$$

with inner product

$$(u_1, u_2) = \int_M u_1(x) \overline{u_2(x)} d\mu(x).$$

The space $L^2(M)$ decomposes into a countable direct sum of mutually orthogonal subspaces $\{V_k\}_{k=0,1,\dots}$ called (generalized) spherical harmonics. Let $\{\phi_{k,i}\}_{i=1}^{h_k}$ be an orthonormal basis for V_k , where $h_k = \dim V_k$. Since M is distance transitive, the function

$$\Phi_k(x, y) := \sum_{i=1}^{h_k} \phi_{k,i}(x) \overline{\phi_{k,i}(y)}$$

depends only on $\tau(x, y)$. This expression is called the addition formula, and $\Phi_k(\tau)$ is called the zonal spherical function associated with V_k . It is immediate

from the definition that Φ_k is positive definite, that is,

$$\sum_{x \in X} \sum_{y \in X} \Phi_k(\tau(x, y)) \geq 0$$

for any $X \subset M$. Throughout this paper, we assume that $\{\Phi_i\}$ forms a family of orthogonal polynomials. Remark that for all known two-point-homogeneous spaces, Φ_i are polynomials. We suppose that the degree of Φ_k is k . Note that $\Phi_k(\tau_0) = h_k$, where $\tau_0 = \tau(x, x)$ for $x \in X$.

Example 2.1. The unit sphere $S^{m-1} \subset \mathbb{R}^m$ is a two-point-homogeneous space. A polynomial with m variables is said to be harmonic if its Laplacian is equal to zero. Then, V_k is the linear space of all homogeneous harmonic polynomials of degree k , with m variables, and $h_k = \binom{m+k-1}{k} - \binom{m+k-3}{k-2}$. The polynomial Φ_k is the Gegenbauer polynomial $G_k(x)$. The Gegenbauer polynomials G_k are defined by the following manner:

$$xG_k(x) = \lambda_{k+1}G_{k+1}(x) + (1 - \lambda_{k-1})G_{k-1}(x)$$

where $\lambda_k = k/(m + 2k - 2)$, $G_0(x) \equiv 1$, and $G_1(x) = mx$.

2.2 Distance sets and t -designs

Let M be a two-point-homogeneous space. We define

$$A(X) = \{\tau(x, y) \mid x, y \in X, x \neq y\}$$

for a finite set X in M . A finite $X \subset M$ is called an s -distance set (or s -code) if $|A(X)| = s$. If X is an s -distance set, then $|X| \leq \sum_{i=0}^s h_i$ [7].

A finite $X \subset M$ is called a t -design if

$$\sum_{x, y \in X} \Phi_i(\tau(x, y)) = 0$$

for any $1 \leq i \leq t$. We say that X has strength t if X is a t -design but not a $(t+1)$ -design. It is proved by the same method in [7] that if X is a 2e-design, then $|X| \geq \sum_{i=0}^e h_i$.

Let $\{\phi_{k,i}\}_{i=1}^{h_k}$ be an orthonormal basis of V_k . Let H_k be the matrix whose (i, j) -entry is $\phi_{k,j}(x_i)$, where $X = \{x_1, x_2, \dots, x_n\} \subset M$. The matrix H_k is called the characteristic matrix of degree k .

The following are needed later.

Theorem 2.2 ([12]). *The product of any two zonal spherical functions $\Phi_i(t)$ and $\Phi_j(t)$ can be expressed as*

$$\Phi_i(t)\Phi_j(t) = \sum_{k=0}^{i+j} c_{i,j}^k \Phi_k(t)$$

with $c_{i,j}^k \geq 0$ and $c_{i,j}^0 = h_i \delta_{i,j}$.

Lemma 2.3. *Let $F(t) = \sum_k f_k \Phi_k(t)$, and $G(t) = \Phi_l(t)F(t)/h_l = \sum_k g_k \Phi_k(t)$. Then $g_0 = f_l$.*

Proof. By Theorem 2.2,

$$\sum_k g_k \Phi_k(t) = \frac{\Phi_l(t)}{h_l} \sum_k f_k \Phi_k(t) = \frac{1}{h_l} \sum_k f_k \sum_{i=0}^{k+l} c_{i,k}^i \Phi_i(t).$$

Since $c_{i,j}^0 = h_i \delta_{i,j}$, this lemma follows. \square

Define $\|N\|^2 = \sum_{i,j} n_{i,j}^2$, where $n_{i,j}$ is the (i,j) -entry of a matrix N . Let ${}^t N$ be the transpose of a matrix N .

Theorem 2.4 ([7]). *Let $X \subset M$ and $F(t) = \sum_{k=0}^{\infty} f_k \Phi_k(t)$. If $F(\alpha) = 0$ for any $\alpha \in A(X)$ and $F(\tau_0) = 1$, then*

$$|X|(1 - |X|f_0) = \sum_{k=1}^{\infty} f_k \|{}^t H_k H_0\|^2.$$

Theorem 2.5 ([7]). *If X is a t -design in M , then for nonnegative integers k, l such that $k + l \leq t$,*

$${}^t H_k H_l = |X| \Delta_{k,l},$$

where $\Delta_{k,l}$ is the identity matrix if $k = l$, and $\Delta_{k,l}$ is the zero matrix if $k \neq l$.

We define

$$F_X(t) := \prod_{\alpha \in A(X)} \frac{t - \alpha}{\tau_0 - \alpha}.$$

Lemma 2.6. *Let X be an s -distance set with strength t in M . We have $F_X(t) = \sum_{k=0}^s f_k \Phi_k(t)$. If $t \geq s - 1$, then $f_{t-s+1} \neq 1/|X|$.*

Proof. Let $G(t) = \Phi_{t-s+1}(t) F_X(t) / h_{t-s+1} = \sum_{k=0}^{t+1} g_k \Phi_k(t)$. By Theorem 2.4,

$$|X|(1 - |X|g_0) = \sum_{k=1}^{t+1} g_k \|{}^t H_k H_0\|^2,$$

where $g_{t+1} \neq 0$. By Theorem 2.5, $|X|(1 - |X|g_0) = g_{t+1} \|{}^t H_{t+1} H_0\|^2$. Since X is not a $(t+1)$ -design, $\|{}^t H_{t+1} H_0\|^2 \neq 0$ and hence $1 - |X|g_0 \neq 0$. By Lemma 2.3, $f_{t-s+1} = g_0$ and $f_{t-s+1} \neq 1/|X|$. \square

2.3 Cometric association schemes

Let X be a finite set and $\mathcal{R} = \{R_0, R_1, \dots, R_s\}$ be a set of non-empty subsets of $X \times X$. For $0 \leq i \leq s$, let A_i be the $(0,1)$ -matrix indexed by the elements of X , whose (x,y) -entry is 1 if $(x,y) \in R_i$, and 0 otherwise. The matrix A_i is called the adjacency matrix of the graph (X, R_i) . A pair (X, \mathcal{R}) is a symmetric association scheme of class s if the following hold:

- (1) A_0 is the identity matrix;
- (2) $\sum_{i=0}^s A_i = J$, where J is the all one matrix;
- (3) ${}^t A_i = A_i$ for $1 \leq i \leq s$;
- (4) $A_i A_j$ is a linear combination of A_0, A_1, \dots, A_s for $0 \leq i, j \leq s$.

The vector space \mathcal{A} spanned by A_i over the real field \mathbb{R} is an algebra which is called the Bose-Mesner algebra of (X, \mathcal{R}) . Since \mathcal{A} is semi-simple and commutative, there exist primitive, mutually orthogonal idempotents $\{E_0, E_1, \dots, E_s\}$ where $E_0 = \frac{1}{|X|}J$. Let m_i be the rank of E_i for $0 \leq i \leq s$. Since $\sum_{i=0}^s E_i = I$ and $\{E_0, E_1, \dots, E_s\}$ are mutually orthogonal idempotents,

$$\sum_{i=0}^s m_i = |X|. \quad (2.1)$$

Since \mathcal{A} is closed under the entry-wise product \circ , we define the Krein parameters $q_{i,j}^k$ by

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^s q_{i,j}^k E_k. \quad (2.2)$$

The Krein parameters are nonnegative real numbers [1, Theorem 3.8] [16]. The scheme (X, \mathcal{R}) is Q -polynomial (or cometric) with respect to the ordering E_0, E_1, \dots, E_s if the following hold: $q_{1,j}^k > 0$ if $k = j \pm 1$ and $q_{1,j}^k = 0$ if $k < j - 1$ or $j + 1 < k$. If (X, \mathcal{R}) is Q -polynomial, for notational convenience, set $a_i^* = q_{1,i}^i$ ($0 \leq i \leq s$), $b_i^* = q_{1,i+1}^i$ ($0 \leq i \leq s - 1$), and $c_i^* = q_{1,i-1}^i$ ($1 \leq i \leq s$), $c_0^* = b_s^* = 0$, we abbreviate m_1 as m . The matrix $B_1^* = (q_{1,j}^k)_{0 \leq j, k \leq s}$ is said to be the Krein matrix. It follows from [1, Proposition 3.7] that $a_0^* = 0$, $c_1^* = 1$ and

$$a_i^* + b_i^* + c_i^* = m \text{ for } 0 \leq i \leq s, \quad (2.3)$$

$$b_i^* m_i = c_{i+1}^* m_{i+1} \text{ for } 0 \leq i \leq s - 1. \quad (2.4)$$

Then (2.2) gives the following three-term recurrence: $E_1 \circ E_i = \frac{1}{|X|} (c_{i+1}^* E_{i+1} + a_i^* E_i + b_{i-1}^* E_{i-1})$ for $1 \leq i \leq s - 1$. Denote $A^{\circ i}$ by the product $A \circ A \circ \dots \circ A$ to i factors. The following lemma is used in a proof of Theorem 4.1:

Lemma 2.7 ([1, Lemma 4.7]). *Let A be a square matrix of rank m over real field \mathbb{R} . The following hold:*

- (1) $\text{rank } A^{\circ h} \leq \binom{m+h-1}{h}$ for any nonnegative integer h .
- (2) If the equality in (1) holds for some h , then the equality in (1) holds for any $j \leq h$.

Proof. (1): Let $\{a_1, \dots, a_m\}$ be a basis of the vector space spanned by the rows of A . Since the row space of $A^{\circ h}$ are spanned by a set $\{a_{i_1} \circ \dots \circ a_{i_h} \mid 1 \leq i_1 \leq \dots \leq i_h \leq m\}$, the desired result follows.

(2): Suppose that $\text{rank } A^{\circ h} = \binom{m+h-1}{h}$ for some nonnegative integer h . This is equivalent that the set $\{a_{i_1} \circ \dots \circ a_{i_h} \mid 1 \leq i_1 \leq \dots \leq i_h \leq m\}$ is linearly independent. Then a set $\{a_{i_1} \circ \dots \circ a_{i_j} \mid 1 \leq i_1 \leq \dots \leq i_j \leq m\}$ is also linearly independent for any $j \leq h$. Indeed, for $j \leq h$, assume $\sum c_{i_1, \dots, i_j} a_{i_1} \circ \dots \circ a_{i_j} = 0$ for some $c_{i_1, \dots, i_j} \in \mathbb{R}$, where indices run through $1 \leq i_1 \leq \dots \leq i_j \leq m$. Multiplying $a_1^{\circ(h-j)}$, we have $\sum c_{i_1, \dots, i_j} a_1^{\circ(h-j)} \circ a_{i_1} \circ \dots \circ a_{i_j} = 0$. Since the set $\{a_{i_1} \circ \dots \circ a_{i_h} \mid 1 \leq i_1 \leq \dots \leq i_h \leq m\}$ is linear independent, $c_{i_1, \dots, i_j} = 0$ for all indices. Therefore the desired result is proved. \square

We can consider the embedding of a Q -polynomial association scheme into the unit sphere as follows. Since the primitive idempotent E_1 is positive semi-definite, there exists a $|X|$ times m matrix U of rank m such that $\frac{|X|}{m} E_1 = U^t U$.

Since E_1 has no repeated rows, U has also no repeated rows. Corresponding x in X to the x -th row vector of U , we identify X as the row vectors U . Then X is always a spherical 2-design. X is a spherical 3-design if and only if $a_1^* = q_{1,1}^1 = 0$. Further, we have a characterization for X to be a spherical t -design in terms of the Krein parameters as follows:

Theorem 2.8 ([18, Theorem 3.1]). *Let (X, \mathcal{R}) be a Q -polynomial scheme. Then the following are equivalent:*

- (1) X is a spherical t -design.
- (2) $a_i^* = 0$ for $0 \leq i \leq \lfloor (t-1)/2 \rfloor$ and $c_j^* = \frac{m_j}{m+2j-2}$ for $0 \leq j \leq \lceil (t-1)/2 \rceil$.

3 Bounds on s -distance sets with strength t

Let $D_i = H_i^t H_i$, and $\mathcal{E}_+^{(i)}$ denote the direct sum of the eigenspaces corresponding to the all positive eigenvalues of D_i .

Lemma 3.1. *The inequality $\dim \mathcal{E}_+^{(i)} \leq h_i$ holds.*

Proof. Since D_i is positive semidefinite, the rank of D_i is equal to $\dim \mathcal{E}_+^{(i)}$. Note that the rank of H_i is at most h_i . Therefore the rank of D_i is at most h_i . \square

Let $\mathcal{E}_0^{(i)}$ denote the eigenspace corresponding to the zero eigenvalue of D_i . For each $x \in X$, let e_x be the column vector, whose x -th entry is 1, and other entries are 0. Let V denote the real vector space spanned by $\{e_x\}_{x \in X}$. Since D_i is a positive semidefinite matrix, $V = \mathcal{E}_+^{(i)} \oplus \mathcal{E}_0^{(i)}$ for each i .

Lemma 3.2. *Let X be an s -distance set in M . Suppose we have $F_X(t) = \sum_{k=0}^s f_k \Phi_k(t)$, where f_k are real numbers. Then*

$$V = \sum_{k: f_k > 0} \mathcal{E}_+^{(k)}.$$

Proof. Suppose there exists $v \notin \sum_{k: f_k > 0} \mathcal{E}_+^{(k)}$. Then, we can write $v = v_1 + v_2$, where $v_1 \in \sum_{k: f_k > 0} \mathcal{E}_+^{(k)}$ and $0 \neq v_2 \in \bigcap_{k: f_k > 0} \mathcal{E}_0^{(k)}$. Note that $I = \sum_{k=0}^s f_k D_k$. Therefore,

$$\begin{aligned} v_2 &= \sum_{k=0}^s f_k D_k v_2 \\ &= \sum_{k: f_k < 0} f_k D_k v_2. \end{aligned}$$

Then, the matrix $\sum_{k: f_k < 0} f_k D_k$ has an eigenvalue 1. This contradicts that $\sum_{k: f_k < 0} f_k D_k$ is negative semidefinite. \square

Lemma 3.3. *Let X be an s -distance set with strength $2s - i$ in M , where $2 \leq i \leq 2s$. Suppose $F_X(t) = \sum_{k=0}^s f_k \Phi_k(t)$, where f_i are real numbers, and $f_j \neq 1/|X|$ for some $\max\{s - i + 1, 0\} \leq j \leq \lfloor (2s - i)/2 \rfloor$. Then, we have $\mathcal{E}_+^{(j)} \subset W$, where*

$$W = \sum_{k=\lfloor s - \frac{i}{2} \rfloor + 1}^s \mathcal{E}_+^{(k)}.$$

Proof. By Theorem 2.5,

$$\begin{aligned} D_j &= \sum_{k=0}^s f_k D_k D_j \\ &= f_j |X| D_j + \sum_{k=\lfloor s-\frac{i}{2} \rfloor + 1}^s f_k D_k D_j, \end{aligned}$$

for $\max\{s-i+1, 0\} \leq j \leq \lfloor (2s-i)/2 \rfloor$. Therefore,

$$(1 - f_j |X|) D_j = \sum_{k=\lfloor s-\frac{i}{2} \rfloor + 1}^s f_k D_k D_j.$$

where $1 - f_j |X| \neq 0$. Let v be an eigenvector corresponding to an eigenvalue $\lambda > 0$ of D_j . We can write $v = \sum_m v_m^{(k)}$ for each k , where $v_m^{(k)}$ is an eigenvector corresponding to an eigenvalue $\lambda_m^{(k)}$ of D_k . Then,

$$\begin{aligned} (1 - f_j |X|) D_j v &= \sum_{k=\lfloor s-\frac{i}{2} \rfloor + 1}^s f_k D_k D_j v \\ (1 - f_j |X|) \lambda v &= \lambda \sum_{k=\lfloor s-\frac{i}{2} \rfloor + 1}^s f_k D_k v \\ &= \lambda \sum_{k=\lfloor s-\frac{i}{2} \rfloor + 1}^s f_k D_k \sum_m v_m^{(k)} \\ &= \lambda \sum_{k=\lfloor s-\frac{i}{2} \rfloor + 1}^s f_k \sum_m \lambda_m^{(k)} v_m^{(k)}. \end{aligned} \tag{3.1}$$

Note that eigenvectors corresponding to the zero eigenvalue vanishes in (3.1). Hence we have $v \in W$. The set of eigenvectors corresponding to positive eigenvalues of D_j is a basis of $\mathcal{E}_+^{(j)}$. Therefore, $\mathcal{E}_+^{(j)} \subset W$. \square

Remark 3.4. Suppose the conditions in Lemma 3.3. When $s-i \geq 0$, we have $f_j = 1/|X|$ for $0 \leq j \leq s-i$ [7].

The following is the main theorem in this section.

Theorem 3.5. Let X be an s -distance set with strength $2s-i$ in M , where $2 \leq i \leq 2s$. Suppose $F_X(t) = \sum_{k=0}^s f_k \Phi_k(t)$, where f_k are real numbers. Then,

$$|X| \leq \sum_{k=0}^{s-i} h_k + \sum_{\substack{k=\max\{s-i+1, 0\} \\ k: f_k = \frac{1}{|X|}}}^{\lfloor s-\frac{i}{2} \rfloor} h_k + \sum_{k=\lfloor s-\frac{i}{2} \rfloor + 1}^s h_k,$$

where $\sum_{k=0}^{s-i} h_k = 0$ if $s-i < 0$.

Proof. By Remark 3.4 and Lemmas 3.1, 3.2, and 3.3,

$$\begin{aligned}
|X| &= \dim\left(\sum_{k:f_k>0} \mathcal{E}_+^{(k)}\right) \\
&\leq \dim\left(\sum_{k=0}^s \mathcal{E}_+^{(k)}\right) \\
&= \dim\left(\sum_{k=0}^{s-i} \mathcal{E}_+^{(k)} \oplus \sum_{\substack{k=\max\{s-i+1,0\} \\ k:f_k=\frac{1}{|X|}}}^{\lfloor s-\frac{i}{2} \rfloor} \mathcal{E}_+^{(k)} \oplus \sum_{k=\lfloor s-\frac{i}{2} \rfloor+1}^s \mathcal{E}_+^{(k)}\right) \\
&\leq \sum_{k=0}^{s-i} h_k + \sum_{\substack{k=\max\{s-i+1,0\} \\ k:f_k=\frac{1}{|X|}}}^{\lfloor s-\frac{i}{2} \rfloor} h_k + \sum_{k=\lfloor s-\frac{i}{2} \rfloor+1}^s h_k.
\end{aligned}$$

□

Corollary 3.6. *Let X be an s -distance set with strength $2s - i$ in M , where $2 \leq i \leq s + 1$. Then,*

$$|X| \leq \sum_{k=0}^s h_k - h_{s-i+1}.$$

Proof. By Lemma 2.6, $f_{s-i+1} \neq 1/|X|$. By Theorem 3.5, this corollary follows. □

Remark 3.7. If we prove $f_k \neq 1/|X|$ for some $s - i + 2 \leq k \leq \lfloor s - \frac{i}{2} \rfloor$ under the assumption in Corollary 3.6, then the upper bound is improved.

A finite $X \subset S^{m-1}$ is said to be antipodal if $-x \in X$ for any $x \in X$. Let $\delta_s = 1$ if s is odd, and $\delta_s = 0$ if s is even.

Corollary 3.8. *Let X be an antipodal s -distance set with strength $2s - 2i - 1$ in S^{m-1} , where $1 + \delta_s \leq i \leq s + \delta_s$. Then,*

$$|X| \leq 2 \sum_{k=0}^{\frac{s-\delta_s}{2}} h_{2k} - 2h_{s+\delta_s-2i}.$$

Proof. The finite set X is identified with a $|X|/2$ point $((s - \delta_s)/2)$ -distance set with strength $s - i - 1$ in the real projective space [13, Theorem 9.2]. Note that the real projective space is a two-point-homogeneous space. Let h_k be the dimension of the spherical harmonics for S^{m-1} , and \bar{h}_k be that for the real projective space. By Corollary 3.6 for the real projective space,

$$\frac{|X|}{2} \leq \sum_{k=0}^{\frac{s-\delta_s}{2}} \bar{h}_k - \bar{h}_{\frac{s}{2} + \frac{\delta_s}{2} - i}$$

for $1 + \delta_s \leq i \leq s + \delta_s$. Note that $\bar{h}_k = h_{2k}$ [13]. Therefore, this corollary follows. □

4 Bounds on Q -polynomial schemes

Theorem 4.1. *Let (X, \mathcal{R}) be an s -class Q -polynomial scheme with respect to the ordering E_0, E_1, \dots, E_s . Define $l = \max\{k \in \{0, 1, \dots, s\} \mid a_0^* = \dots = a_k^* = 0\}$.*

(1) *If $l = s$ holds, then $|X| \leq 2 \binom{m+s-2}{s-1}$. The equality holds if and only if X is a $(2s-1)$ -design and $m_i = h_i$ holds for $0 \leq i \leq s-1$ and $m_s = \binom{m+s-2}{s-1} - \binom{m+s-3}{s-2}$.*

(2) *If $(s-1)/2 \leq l \leq s-1$ holds, then $|X| \leq \binom{m+2l-s}{2l+1-s} + \binom{m+s-1}{s}$ holds. The equality holds if and only if*

$$m_i = \begin{cases} h_{i,m} & \text{if } 2 \leq i \leq l+1, \\ \sum_{k=0}^{i-l-2} (h_{i-2k,m} - h_{i-2k-1,m}) & \text{if } l+2 \leq i \leq s. \end{cases}$$

(3) *If $l \leq (s-2)/2$ holds, then $|X| \leq \binom{m+s-1}{s}$ holds. The equality holds if and only if*

$$m_i = \begin{cases} h_{i,m} & \text{if } 2 \leq i \leq l+1, \\ \sum_{k=0}^{i-l-2} (h_{i-2k,m} - h_{i-2k-1,m}) & \text{if } l+2 \leq i \leq 2l+2, \\ \binom{m+i-1}{i} - \binom{m+i-2}{i-1} & \text{if } 2l+3 \leq i \leq s. \end{cases}$$

Moreover, when the equality holds in each case (2) or (3), X is a spherical $(2l+2)$ -design.

Proof. Suppose $l = s$, namely the scheme (X, \mathcal{R}) is Q -bipartite. Then, by [15, Corolary 4.2], the image of the embedding of the scheme into the unit sphere is an antipodal set. Therefore it follows from [6] that $|X| \leq 2 \binom{m+s-2}{s-1}$. When the equality holds, [7, Theorem 6.8, Remark 7.6] says that $m_i = Q_i(1) = h_i$ holds for $0 \leq i \leq s-1$ and by (2.4), $m_s = \binom{m+s-2}{s-1} - \binom{m+s-3}{s-2}$.

Suppose $l \leq s-1$. The three term recurrence and the conditions $a_1^* = \dots = a_l^* = 0$ implies that for $0 \leq i \leq l+1$, there exist positive real numbers $f_{i,k}$ such that

$$E_1^{\circ i} = \sum_{\substack{k=0 \\ k \equiv i \pmod{2}}}^i f_{i,k} E_k. \quad (4.1)$$

When $l+1 \leq s-1$, the three term recurrence and $a_{l+1}^* \neq 0$ imply that for $1 \leq i \leq s-l-1$, there exist positive real numbers $f_{l+1+i,k}$ such that

$$E_1^{\circ(l+1+i)} = \sum_{\substack{k=0 \\ k \equiv l+1+i \pmod{2}}}^{l+1+i} f_{l+1+i,k} E_k + \sum_{\substack{k=\max\{l+2-i, 0\} \\ k \equiv l+i \pmod{2}}}^{l+i} f_{l+1+i,k} E_k. \quad (4.2)$$

Taking the rank of the both hand sides in (4.1) and (4.2), it follows from

Lemma 2.7 that

$$\sum_{\substack{k=0 \\ k \equiv i \pmod{2}}}^i m_k \leq \binom{m+i-1}{i} \text{ for } 0 \leq i \leq l+1, \quad (4.3)$$

$$\sum_{\substack{k=0 \\ k \equiv l+1+i \pmod{2}}}^{l+1+i} m_k + \sum_{\substack{k=\max\{l+2-i, 0\} \\ k \equiv l+i \pmod{2}}}^{l+i} m_k \leq \binom{m+l+i}{l+i+1} \text{ for } 1 \leq i \leq s-l-1. \quad (4.4)$$

(2): Substituting $i = 2l+1-s$ and $i = s-l-1$ into (4.3) and (4.4) respectively, using the equation (2.4),

$$\begin{aligned} |X| &= \sum_{k=0}^s m_k \\ &\leq \sum_{\substack{k=0 \\ k \equiv 2l+1-s \pmod{2}}}^{2l+1-s} m_k + \sum_{\substack{k=0 \\ k \equiv s \pmod{2}}}^s m_k + \sum_{\substack{k=2l+3-s \\ k \equiv s-1 \pmod{2}}}^{s-1} m_k \\ &\leq \binom{m+2l-s}{2l+1-s} + \binom{m+s-1}{s}. \end{aligned}$$

The equalities hold in case (2) if and only if $\text{rank } E_1^{o(2l-s)} = \binom{m+2l-s}{2l+1-s}$ and $\text{rank } E_1^{os} = \binom{m+s-1}{s}$. By Lemma 2.7 (2), this condition is equivalent to $\text{rank } E_1^{oi} = \binom{m+i-1}{i}$ for $0 \leq i \leq s$. Then we obtain a system of linear equations from (4.3) for $2 \leq i \leq l+1$ and (4.4) for $1 \leq i \leq s-l-1$ whose unknowns are $\{m_i \mid 2 \leq i \leq s\}$. Its coefficient matrix is a lower triangular matrix with non-zero diagonals. Therefore the equality holds if and only if m_i for $2 \leq i \leq s$ are uniquely determined as desired.

(3): Substituting $i = s-l-1$ into (4.4), using the equation (2.4),

$$\begin{aligned} |X| &= \sum_{k=0}^s m_k \\ &= \sum_{\substack{k=0 \\ k \equiv s \pmod{2}}}^s m_k + \sum_{\substack{k=0 \\ k \equiv s-1 \pmod{2}}}^{s-1} m_k \\ &\leq \binom{m+s-1}{s}. \end{aligned}$$

Since the same method in (2) is applied in (3), the equality holds if and only if m_i for $2 \leq i \leq s$ are uniquely determined as desired.

When the equality holds in each case (2), (3), $m_i = h_{i,m}$ holds for $0 \leq i \leq l+1$. Repeated application of the formula (2.4) together with (2.3), we have $c_j^* = mj/(m+2j-2)$ for $0 \leq j \leq l+1$. Recall we assume $a_i^* = 0$ for $0 \leq i \leq l$. It follows from Theorem 2.8 that X is a spherical $(2l+2)$ -design. \square

Remark 4.2. In Theorem 4.1 when the equality holds for $l = s-1$ (resp. $l = s$), then X is a tight $2s$ -design (resp. tight $(2s-1)$ -design) in S^{m-1} [7].

5 Examples

Example 5.1. Let Ω be the minimum vectors, which rescaled to the norm 1, of the Leech lattice in \mathbb{R}^{24} . Fix $u, v \in \Omega$ such that $(u, v) = -1/4$. Define X by $\{x \in \Omega \mid (x, u) = 1/2, (x, v) = 0\}$. Then $|X| = 2025$. Considering the projection onto \mathbb{R}^{22} , we may regard X as a subset in the S^{21} . Then X is a spherical 3-distance set with strength 4 in S^{21} . Since $h_0 = 1$, $h_1 = 22$, and $h_3 = 2002$, X attains the upper bound in Corollary 3.6.

On the other hand, since X satisfies $t = 2s - 2$, X carries a Q -polynomial scheme whose Krein matrix B_1^* is

$$B_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 22 & 0 & 11/6 & 0 \\ 0 & 21 & 27/22 & 30/11 \\ 0 & 0 & 625/33 & 212/11 \end{pmatrix}.$$

Then the scheme X also attains the bound in Theorem 4.1.

Example 5.2. Let X be the set of vertices of the dodecahedron in \mathbb{R}^3 . Then, X is an antipodal spherical 5-distance set with strength 5. Since $h_0 = 1$ and $h_4 = 9$, X attains the upper bound in Corollary 3.8.

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